Portfolio Selection with Randomly Time-Varying Moments: The Role of the Instantaneous Capital Market Line

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Abstract

In the intertemporal portfolio selection model of Merton (1973), any change in means, variances or covariances of security returns is sufficient to generate a change in the investment opportunity set. Merton's formulation suggests that investors will hedge all such changes by including in their optimal portfolio holdings as many hedge portfolios as there are state variables that describe the dynamics of returns. In this paper, we show that investors need to hedge only against changes in the random slope and position of the instantaneous capital market line. If the instantaneous capital market line is constant or deterministic, then investors will not hold any hedge portfolios at all, even though means, variances and covariances of securities returns may be changing randomly over time. Based on these results, we propose a new definition of the investment opportunity set and changes in the investment opportunity set. Our analysis allows for incomplete markets and does not assume that the securities prices are Markovian. It provides a potential theoretical foundation for certain conditional tests of asset pricing models which ignore the intertemporal hedging premia.

JEL classification: G11, G12

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1 Introduction

This paper re-examines the role of hedge portfolios and the definition and nature of the "investment opportunity set" in intertemporal portfolio selection.

According to Merton (1973) "a sufficient set of statistics for the [investment] opportunity set at a given point in time" is the means, standard deviations and correlations of the instantaneous rates of return to all the securities in the model. Furthermore, "the dynamics for the changes in the opportunity set over time" are given by a set of Itô processes that describe changes in the instantaneous means and standard deviations and potentially in the correlations, and which, together with the securities price processes, form a Markovian vector of state variables[1].

This definition implies that any change in means, variances or covariances generates a change in the investment opportunity set. Merton's formulation, as well as the formulation in later papers such as Cox and Huang (1989), suggests that investors will hedge all such changes by including in their optimal portfolios as many hedge portfolios ("hedge funds" in Merton's terminology) as there are state variables that describe the dynamics of the returns.

We show that investors will only hedge against changes in instantaneous means, variances and covariances of returns that affect the slope or position of the instantaneous capital market line (ICML). They do not need to hedge against changes in first and second moments which do not affect the intercept or the slope of the ICML.

We conclude from these results that the ICML rather than all the means, standard deviations, and correlations of instantaneous rates of return should define the investment opportunity set. This dramatically reduces the dimensionality of the portfolio selection problem.

These results are similar in spirit to those of Constantinides (1980). He identifies some circumstances in which investors will not hold hedge portfolios even though some asset returns may be non-stationary, and in particular, even though some assets may have stochastically time-varying means, vari-

[1]see page 483 in Merton (1992)

ances, and correlations. Specifically, he shows that this is true in equilibrium if the investors' utility functions have the aggregation property and all assets in postive supply have stationary returns. Our results do not assume equilibrium or aggregating utility functions, and our assumptions about returns concern only the dynamics of the ICML.

The capital market line (CML) has always played a central role in static mean-variance portfolio theory. It would be graphed and used to visualize the determination of the optimal portfolio. Merton (1971) showed that if the interest rate and all the first and second moments are constant, then the ICML has the same role in continuous time as the CML has in a static model. However, the ICML has played no role in continuous time when moments vary stochastically.

This created a dichotomy between the way we understood portfolio theory in discrete and continuous time. Our results imply that the ICML plays the role of the investment opportunity set also when the moments vary stochastically over time.

Expressed in terms of the ICML, portfolio analysis in continuous time is as intuitive as it is in discrete time. If the investment opportunity set is constant or deterministic, then the investor will place himself along the ICML, even if the moments of the security returns change randomly over time. He will slide up and down the ICML over time as his wealth and his risk aversion coefficient change. If not only the moments of the security returns but also the intercept and slope of the ICML change randomly over time, then the investor will typically hedge against those changes. To hedge against a potential deterioration of the investment opportunity set, such as a decline in the interest rate or in the slope of the ICML, he holds one or more hedge portfolios which tend to do well in these scenarios. The opportunity cost that he incurs is reflected in a deviation from instantaneous mean-variance efficiency. In other words, he pays for the hedge portfolios by placing himself below the ICML.

Our results can alternatively be stated in terms of the drift and volatility of the state price process, because the drift and volatility are identical to the (negative of the) intercept and the slope of the ICML. Investors hedge only against changes in instantaneous means, variances and covariances that affect the drift or volatility of the state price process. If the drift and volatility of the state price process are deterministic, then investors do not need to hedge at all. Thus, our analysis adds yet another aspect to the role of the state price process.

Our results are derived within a model where markets may be dynamically incomplete because the number of sources of uncertainty may be larger than the number of risky assets[2]. This is the type of market incompleteness analyzed by He and Pearson (1991) and Karatzas et al. (1991). In general, an optimal portfolio strategy may not even exist in such a model. However, we are able to give plausible conditions for existence, and we can characterize the optimal strategy. The expressions for the optimal portfolios in incomplete markets are the same as those in complete markets. This greatly simplifies the analysis of this type of incomplete markets.

Our analysis also has some implications for empirical asset pricing. First, given our new definition of the investment opportunity set, one should be cautious in interpreting empirical evidence of predictability as implying that the investment opportunity set is changing over time. Security returns and their moments may well be predictable while the investment opportunity set is constant.

Secondly, our analysis provides a theoretical justification for ignoring the intertemporal hedging premia in conditional asset pricing tests, when these explicitly or implicitly assume a constant ICML.

Our modeling approach is closely related to an optimal portfolio selection result of Chamberlain (1988). We follow Chamberlain in assuming that the state variables form a Wiener process. However, Chamberlain's other assumptions and their economic interpretation differ from ours. He did not make assumptions about hedge portfolios or about the slope of the ICML. He assumed that the interest rate is zero, that every claim which is a function of the final value of the state price process is marketed, and that this final value is a function of the paths of the state variable. Furthermore, Chamberlain did not express his optimal portfolio selection result (in his Lemma 3) in terms of funds but in terms of martingales and stochastic integrals: "the value of an optimal portfolio is restricted to be a stochastic integral over a single (vector) martingale [...], which is common to all optimal portfolios."

[2]Note that this type of market incompleteness differs from that which results from non-traded assets or stochastic labor income.

Thus the economic content of Chamberlain's paper is different from ours, even though the mathematics is similar. Chamberlain was not concerned with the capital market line or the investment opportunity set, which are the crucial concepts in the present paper. He motivated his assumptions by relating them to factor models and the APT. These ideas do not play any role here.

Our economic results also rely on some modeling innovations. First, we find that in order to express the optimal portfolios in terms of intertemporal hedge portfolios, it is not necessary to assume that the interest rate and the mean vector and dispersion matrix of the instantaneous returns are functions of a vector of state variables, as in Merton (1973), Cox and Huang (1989), and the subsequent literature. It suffices to assume that the interest rate and the slope of the ICML satisfy such a condition. Moreover, they need only be functions of the paths, not the levels, of the state variables.

Secondly, to get the hedging portfolios to disappear, it is not necessary to assume that the interest rate and the mean vector and dispersion matrix of the instantaneous returns are deterministic. It is not even necessary to assume that the interest rate and the vector of prices of risk are deterministic. It suffices to assume that the interest rate and the slope of the ICML are deterministic.

Ocone and Karatzas (1991) show that if markets are complete, and if the interest rate and the vector of prices of risk are deterministic, then investors do not need to hedge. However, their assumption that the prices of risk are deterministic is stronger than our assumption that the slope of the ICML is deterministic. Furthermore, their result cannot be recast in terms of the slope of the ICML or the volatility of the state price process, and thus it cannot be used to establish the ICML as the investment opportunity set.

One might conjecture that our results (in the case of complete markets) would follow as a special case from those of Merton (1973) and Cox and Huang (1989). This is not so. Those papers express the holdings of the hedge portfolios in terms of the derivatives with respect to the state variables of an indirect utility function or a value function. To derive our results from theirs, one would need to know that the derivative with respect to a state variable is zero if that state variable does not affect the ICML. However, little is known about the shape of the indirect utility function or the value function, and in particular, there is in general no closed-form expression for its derivatives.

The rest of the study is organized as follows. Section 2 outlines the model. In Section 3 we define the ICML. Section 4 discusses trading and portfolio strategies and the expected utility they generate. Section 5 provides our fund separation result for the case where changes in the position and slope of the ICML are described by a vector of state variables. Section 6 considers the case where the ICML is either deterministic or constant. We conclude in Section 7. A detailed derivation of the results is provided in the appendix.

2 Prices of Risk

We use a standard continuous-times model on a finite time horizon is $[0, T]$. Underlying the model is a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $F = (\mathcal{F}_t)_{t\in[0,T]}$. There are $N + 1$ basic long-lived securities. Security zero is a *money market account* with price process M

$$M(t) = M(0) \exp\left\{\int_0^t r\, ds\right\}$$

where $M(0) > 0$ and r is the *interest rate process*, which is assumed to be measurable, adapted, and pathwise locally integrable with respect to time.

The prices of the remaining N securities are given by an N-dimensional vector S of Itô processes of the form

$$dS = \mathcal{D}(S)\mu\, dt + \mathcal{D}(S)\sigma\, dW$$

where $\mathcal{D}(S)$ is the diagonal matrix with the vector S along the diagonal. The process μ is N-dimensional and is assumed to be measurable, adapted, and pathwise locally integrable with respect to time. The process σ is $(N \times K)$-dimensional and is assumed to be measurable, adapted, and pathwise locally square integrable with respect to time. These processes are not assumed to be functions of a Markovian vector of state variables.

The process W is a K-dimensional Wiener process. Markets may be incomplete, in the sense that there may be many more Wiener processes than there are instantaneously risky securities ($K \gg N$).

A vector of *prices of risk* is a K-dimensional measurable and adapted process λ which is pathwise locally square integrable, such that

$$\mu - r\iota = \sigma\lambda^\top$$

Without loss of generality, we can always choose the so-called *minimal prices of risk*, which are given by

$$\lambda = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1} \sigma$$

With this choice, we find that

$$\begin{aligned} \lambda\lambda^\top &= (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1} \sigma\sigma^\top \left(\sigma\sigma^\top\right)^{-1} (\mu - r\iota) \\ &= (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1} (\mu - r\iota) \end{aligned}$$

Assuming that this process is pathwise locally integrable, λ will be pathwise locally square integrable, as required.

A *state price process* or *pricing kernel* for $\bar{S}$ is a positive one-dimensional Itô process Π such that

$$\frac{d\Pi}{\Pi} = -r\,dt - \lambda\,dW$$

where r is a vector of prices of risk.

3 The Instantaneous Capital Market Line

Recall from mean-variance theory that mean-variance efficient portfolios are portfolios that maximize the expected rate of return given the variance or standard deviation of the rate of return. We can similarly define *instantaneously efficient portfolios* as those that maximize the expected instantaneous rate of return given the standard deviation of the instantaneous rate of return. Their combinations of standard deviation of returns and expected returns plot on a straight line whose intercept with the expected-return axis is the instantaneous interest rate. We call this line the *instantaneous capital market line* (ICML).

It follows from the standard theory that the instantaneously efficient portfolios are the portfolios that are combinations of the money market account and the portfolio $\phi^{\ln}$ given by

$$\phi^{\ln} = \lambda\sigma^\top \left(\sigma\sigma^\top\right)^{-1} = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1}$$

where we note that $\sigma\sigma^\top$ is the covariance matrix of the instantaneous rates of return to the various securities. We call this portfolio the *logarithmic portfolio* because, as is well known and will also follow from the analysis below, it is indeed the optimal portfolio for an investor with a logarithmic utility function.

Given the specific choice we have made for the prices of risk λ,

$$\lambda = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1} \sigma = \phi^{\ln}\sigma$$

The slope of the capital market line is the ratio of excess expected instantaneous rate of return and the standard deviation of the instantaneous rate of return to the logarithmic portfolio. We can calculate this slope as follows.

The excess instantaneous expected rate of return to $\phi^{\ln}$ is

$$\phi^{\ln}(\mu - r\iota) = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1} (\mu - r\iota) = \lambda\lambda^\top$$

The variance of the instantaneous rate of return to $\phi^{\ln}$ is

$$\phi^{\ln}\sigma\sigma^\top\phi^{\ln\top} = \lambda\lambda^\top$$

and the standard deviation is $\sqrt{\lambda\lambda^\top}$. Hence, the slope of the ICML is

$$\frac{\phi^{\ln}(\mu - r\iota)}{\sqrt{\phi^{\ln}\sigma\sigma^\top\phi^{\ln\top}}} = \frac{\lambda\lambda^\top}{\sqrt{\lambda\lambda^\top}} = \sqrt{\lambda\lambda^\top}$$

It follows that the ICML is the straight line with intercept r and slope $\sqrt{\lambda\lambda^\top}$. While the individual elements of the vector λ are prices of risk with respect to the individual Wiener processes, $\sqrt{\lambda\lambda^\top}$ is the price of risk in the aggregate. We can also think of it as the instantaneous Sharpe ratio for instantaneously mean-variance efficient portfolios. It also happens to be the volatility of the state price process.

In the following, we prove that investors will optimally hedge only changes in moments that lead to changes in the slope and position of the ICML. We shall therefore argue that for the purpose of optimal portfolio selection, the instantaneous capital market line is the most useful concept of the "investment opportunity set."

4 Trading and Expected Utility

A *trading strategy* is an adapted and measurable process $\bar{\Delta} = (\Delta_0, \Delta)$ whose values are $(N+1)$-dimensional row vectors. The *value process* of $\bar{\Delta}$ is $\Delta_0 M + \Delta S$.

A trading strategy $\bar{\Delta}$ is *self-financing* if it satisfies the *budget constraint*:

$$\Delta_0(t)M(t) + \Delta(t)S(t) = \Delta_0(0)M(0) + \Delta(0)S(0) + \int_0^t (\Delta_0 \, dM + \Delta \, dS)$$

A *portfolio strategy* is an adapted measurable N-dimensional row vector valued process $\tilde{\theta}$. The interpretation is that $\tilde{\theta}$ tells us the fractions of wealth invested in the various risky securities, while the remaining fraction, $1-\tilde{\theta}\iota$, is invested in the money market account. Here, ι is the N-dimensional column vector all of whose entries are one.

A trading strategy with positive value process can conveniently be expressed as a portfolio strategy. If $\bar{\Delta} = (\Delta_0, \Delta)$ is a self-financing trading strategy such that the value process $V = \bar{\Delta}\bar{S}$ is positive, then the corresponding portfolio strategy is given by

$$\tilde{\Delta} = \Delta\mathcal{D}(S)/V$$

Conversely, we can recover the value process and the trading strategy from knowledge of the portfolio strategy and the initial value $w_0 = \Delta_0 M(0) + \Delta(0)S(0)$ of the trading strategy.

In order to keep things simple, we restrict ourselves to a model with a finite time horizon T and with only final consumption[3].

Let $w_0 > 0$ be the investor's initial wealth level, and let u be his utility function, defined on the positive half-line $(0, \infty)$. If he follows a portfolio strategy $\tilde{\Delta}$, then his expected utility will be $Eu(\bar{\Delta}(T)\bar{S}(T))$, where $\bar{\Delta}$ is the unique self-financing trading strategy corresponding to w_0 and $\tilde{\Delta}$.

5 Fund Separation: Random ICML

We model random changes in the ICML as driven by a number of state variables which are independent Wiener processes relative to the filtration F. We assume that the slope and intercept of the ICML are functions of the paths of the state variables (technically, they are measurable and adapted to the filtration generated by the state variables), but we do not assume that they are functions of the current levels of the state variables[4].

[3]We expect that as is usually the case in models like this, the results can be generalized to a model with a flow of consumption over time and with a finite or an infinite time horizon.

[4]Merton (1973) assumed that the state variables followed a multi-dimensional diffusion process. Our assumption, that they follow an m-dimensional Wiener process, is not as

This way of using state variables is inspired by Chamberlain (1988). Indeed, our proposition below is a version of Chamberlain's mutual fund separation result, but with a different set of assumptions and formulated in terms of the logarithmic portfolio, the hedge portfolios, and the intercept and slope of the ICML[5].

The state variables will be assumed to be hedgeable in the following sense. If B is an m-dimensional Wiener process relative to the filtration F, then we say that B is *hedgeable* if there exists an $(m \times N)$-dimensional measurable and adapted process b such that $b\sigma \in \mathcal{L}^2$, $b\sigma\sigma^\top b^\top = I$, the $m \times m$ identity matrix, and

$$B(t) = \int_0^t b\sigma \, dW$$

We interpret b as a vector of m portfolio strategies which are perfectly instantaneously correlated with the elements of B. We call them the *hedge portfolios* associated with B.

A portfolio strategy $\tilde{\Delta}$ will be said to be a *combination of the hedge portfolios* if it has the form

$$\tilde{\Delta} = \beta b$$

for some $(1 \times m)$-dimensional measurable and adapted process β, called the *coefficient process.*

Proposition *Let B be an m-dimensional Wiener process relative to F. Assume that*

1. *B is hedgeable with associated hedge portfolios b*
2. *The logarithmic portfolio $\phi^{\ln} = (\mu - r\iota)^\top \left(\sigma\sigma^\top\right)^{-1}$ is a combination of the hedge portfolios, with a coefficient process which is measurable and adapted with respect to F^B*

restrictive as it may seem. An m-dimensional diffusion process can, under appropriate regularity conditions, be seen as driven by an m-dimensional Wiener process. If the diffusion coefficient is a continuous function, then the diffusion and the Wiener process generate the same filtration, as shown by Harrison and Kreps (1979).

[5]The case of a deterministic ICML, where $\Pi(T)$ is lognormally distributed, is not covered by Chamberlain's analysis. He assumes that $\Pi(T)$ (or ρ in his notation) is bounded above and below away from zero.

3. *The intercept and slope of the ICML are measurable and adapted with respect to F^B*

Then for each initial wealth level $w_0 > 0$ and each portfolio strategy $\tilde{\Theta}$, there exists a portfolio strategy $\tilde{\Delta}$ which is a combination of the hedge portfolios, such that every risk-averter with initial wealth w_0 will prefer $\tilde{\Delta}$ to $\tilde{\Theta}$.

The proposition implies that if there exists an optimal portfolio strategy for an investor, then there exists one which is a combination of the hedge portfolios. If there is a unique optimal portfolio strategy, then the optimal portfolio strategy is a combination of the hedge portfolios.

There is a detailed proof of the proposition in the appendix. The proof uses the following steps. We build a reduced trading model driven by B and F^B, with the same money market account as the original model and m risky securities with values equal to the values of the hedge portfolios. This reduced model has complete markets. The interest rate and the slope of the ICML are the same in the reduced model as in the original model. Since they are adapted and measurable with respect to F^B, it follows that the state price process is also the same in the reduced model as in the original model. Given the claim Y replicated by $\tilde{\Theta}$, we define a claim Y^* by $Y^* = E(Y \mid \Pi(T))$. This claim is preferred to Y by every risk averter, because it has the same mean as Y but is less risky. It can be replicated within the reduced model, which has complete markets, and hence it can be replicated by a portfolio strategy which is a combination of the hedge portfolios.

Remark: *Suppose the interest rate and the slope of the ICML are functions of the paths of a subset of the state variable processes (including the first of them). Then the remaining state variables are superfluous in the sense that they do not affect the slope and intercept of the ICML and that investors will not hold the associated hedge portfolios. Investors will only hold the money market account, the logarithmic portfolio, and hedge portfolios that hedge against changes in the slope and the position of the ICML.*

Our proposition and its associated remark simplify Merton's separation theorem, since they reveal that it is not necessarily optimal to hedge all changes in the instantaneous means, variances and covariances.

The proposition also leads us to redefine the concept of an investment opportunity set provided in Merton (1973), and consequently, the concept of changes in the investment opportunity set.

Definition: *The investment opportunity set is the instantaneous capital market line. Changes in the investment opportunity set are described by changes in the intercept and slope of the instantaneous capital market line.*

This analysis indicates that the capital market line is as important in continuous time as it is in discrete time.

An attractive special case of the second assumption in the proposition is where the logarithmic portfolio is proportional to one of the hedge portfolios, say the first. Mathematically, this means that there exists a positive process η such that $\phi^{\ln} = \eta b_1$. This process is automatically measurable and adapted with respect to F^B, because

$$\sqrt{\lambda\lambda^\top} = \sqrt{\phi^{\ln}\sigma\sigma^\top\phi^{\ln\top}} = \sqrt{\eta b_1\sigma\sigma^\top b_1^\top\eta} = \sqrt{\eta^2} = \eta$$

Our proposition allows for market incompleteness, because the number of risky securities may be larger than the number of sources of risk. As mentioned earlier, this potential market incompleteness is of exactly the same type as in Karatzas et al. (1991) and He and Pearson (1991). Those papers analyze various conditions for existence of an optimal trading strategy, which is not a simple matter. By contrast, in models with complete markets, existence of an optimal trading strategy is relatively straightforward. It can be ensured by imposing an appropriate integrability condition on the state price process and bounding the investor's relative risk aversion away from zero at high wealth levels.

Our proposition above and the corollary below reveal that the existence of an optimal portfolio strategy is equally straightforward in incomplete markets, provided that the capital market line either is driven by hedgeable state variables or is deterministic. Existence can be ensured by the same conditions on the state price process and the utility function as in a complete market, because these conditions will imply the existence of an optimal portfolio strategy in the reduced trading model constructed from the hedge portfolios, which has complete markets.

Notice that under the assumptions of our proposition, the optimal portfolio holdings will have the same form whether the markets are complete or incomplete.

The proposition does not require that all the instantaneous means, variances and covariances of the securities prices are driven by the state variables. Only the slope and intercept of the ICML are assumed to be driven by the state variables. Furthermore, they do not need to be functions of the state variables, they only need to be functions of their paths. Subject to this requirement, the instantaneous moments of the securities returns can follow general processes.

6 Fund Separation: Deterministic ICML

The following corollary says that when the ICML is deterministic or constant, investors do not at all hedge against changes in the first and second moments of security returns. As in the proposition, the moments do not have to be functions of the levels or the paths of the state variables. They are allowed to vary randomly over time. The only restriction imposed on the dynamics of returns is that the interest rate and the slope of the ICML should be deterministic.

Corollary *Assume that the interest rate is deterministic and that the slope of the ICML is positive and deterministic. Then for each initial wealth level* $w_0 > 0$ *and each portfolio strategy* $\tilde{\Theta}$*, there exists a portfolio strategy* $\tilde{\Delta}$ *which is proportional to the logarithmic portfolio, such that every risk averter with initial wealth* w_0 *will prefer* $\tilde{\Delta}$ *to* $\tilde{\Theta}$*.*

The corollary, like the proposition, implies that if there exists an optimal portfolio strategy for an investor, then there exists one which is a combination of the hedge portfolios. If there is a unique optimal portfolio strategy, then the optimal portfolio strategy is a combination of the hedge portfolios.

The proof of the corollary is provided in the appendix. The idea of the proof is to define a Wiener process B by

$$B(t) = \int_0^t \frac{1}{\sqrt{\lambda\lambda^\top}} \lambda \, dW$$

We show that B is hedgeable with a hedge portfolio which is proportional to the logarithmic portfolio, and then we appeal to the proposition.

The statement that $\tilde{\Delta}$ is proportional to the logarithmic portfolio means that

$$\tilde{\Delta} = \alpha\phi^{\ln}$$

for some one-dimensional measurable and adapted process α. The value weight invested in the logarithmic portfolio is given by α. It can be shown that α equals the relative risk tolerance of the investor's indirect utility function. This follows from the fund separation theorem in Merton (1973) for the case of constant moments, and from the corollary.

Note that the ICML will be constant if and only if the state price process has constant relative drift and volatility. This is so because the state price process has relative drift $-r$ and volatility $\sqrt{\lambda\lambda^\top}$, while the ICML has intercept r and slope $\sqrt{\lambda\lambda^\top}$.

Even with a constant capital market line, the value weight α invested in the logarithmic portfolio will in general change over time in response to changes in the investor's wealth and resultant changes in the risk aversion of his indirect utility function. Changes in α can be interpreted as sliding up and down the ICML.

For an investor with logarithmic utility, the optimal portfolio strategy is the logarithmic portfolio strategy $\phi^{\ln}$. That was why we used that name for it, of course.

In the remainder of this section, we shall discuss the significance of the corollary and the degree to which its assumptions are reasonable.

Apart from (1) the case of logarithmic utility, the only previously known results that we are aware of where all investors will simply invest in the money market account and the logarithmic portfolio are (2) Merton's (1973) example where all moments are constant, (3) Constantinides' (1980) Proposition 1 which assumes aggregating utility functions, equilibrium, and stationary returns to all assets in positive net supply, (4) an example in Karatzas et al. (1991) which assumes power utility, complete markets, a deterministic interest rate r, and a deterministic vector λ of prices of risk; and (5) an example in Ocone and Karatzas (1991), which also assumes complete markets, a deterministic interest rate r, and a deterministic vector λ of prices of risk.

The corollary generalizes all of these results, because it only assumes that the utility function exhibits risk aversion, and it allows σ, μ and λ to change randomly over time, so long as the relation

$$\mu - r\iota = \sigma\lambda^{\top}$$

is satisfied and r and $\lambda\lambda^{\top}$ remain deterministic.

If there are one thousand risky securities, then the case of constant moments requires the constancy of 501,500 parameters, because this is the number of free parameters in the instantaneous excess return vector $\mu - r\iota$ and the instantaneous covariance matrix $\sigma\sigma^{\top}$. The corollary imposes only a two-dimensional restriction on these parameters: the intercept and slope of the ICML should be constant (or equal to deterministic functions of time). In this sense, the corollary has 501,498 more degrees of freedom than the case of constant moments.

A special case of the corollary is where the interest rate and the slope of the ICML are constants. A constant slope $\sqrt{\lambda\lambda^{\top}}$ does not require that the elements of λ stay constant. They may change according to virtually any adapted processes so long as their sum-of-squares is constant. At the same time, all the elements of the matrix σ may change in virtually any non-anticipating way so long as the matrix continues to have full rank. The example below illustrates this. In the example, the elements of λ change randomly because of changes in the vector of instantaneous means, while the elements of σ remain constant.

Example 1 Suppose $N = K$. Suppose the interest rate $r > 0$ and the matrix σ are deterministic constants. Let $x \in \mathbb{R}^K$, $x \neq 0$, and suppose

$$\mu = r\iota + \frac{1}{\|W + x\|}\sigma(W + x)$$

Then

$$\lambda = \frac{1}{\|W + x\|}(W + x)^{\top}$$

and $\lambda\lambda^{\top} = 1$. According to the proposition, the optimal portfolio is proportional to the logarithmic portfolio. Although the instantaneous means vary over time in a highly random manner, investors will not hold hedge portfolios to hedge against these changes. Even so, the composition of their portfolio will change stochastically over time, since we find

$$\phi^{\ln} = \lambda\sigma^{-1} = \frac{1}{\|W + x\|}(W + x)^{\top}\sigma^{-1}$$

□

We can well generate other instances of securities price processes which have a constant ICML. The following example exhibits a class of processes where the second moments are random and the vector λ is constant.

Example 2 Consider any process for the dispersion matrix σ. Let it incorporate any kind of time-varying conditional second moments that may be considered suitable or used in the empirical literature of conditional asset pricing, such as ARCH or GARCH. Let λ be a constant vector, and define the instantaneous mean return processes μ by

$$\mu = r\iota + \sigma\lambda^{\top}$$

or

$$\mu_i - r = \sigma_{i1}\lambda_1 + \sigma_{i2}\lambda_{i2} + \cdots + \sigma_{iK}\lambda_K$$

for each security i. Then the ICML is constant. □

The assumption of a constant ICML is consistent with specifications that have been used or proposed for empirical work. The assumption means that portfolios on the ICML have expected rates of return that are an affine function of their standard deviation. Combined with a partial equilibrium argument, it implies that the market portfolio is on the ICML, and therefore that its expected rate of return is an affine function of its standard deviation.

This is consistent with Merton (1980), who proposed the assumption of a constant capital market line as one of his empirical models for estimating the expected return to the market.

It is also consistent with some of the literature on applications of ARCH-M and GARCH-M processes. These processes model the expected excess rate of return to a security or to the market portfolio as a function of its standard deviation or variance, in addition to modeling the dynamics of the second moments.

For example, French, Schwert and Stambaugh (1987) tested two versions of GARCH-M, where the excess expected rate of return to the market portfolio is an affine function of either the standard deviation or the variance. Bodurtha and Mark (1991) compared three versions of the ARCH-M model, where the excess expected rate of return on the market portfolio is an affine function of either the variance, the standard deviation, or the logarithm of the variance. In both cases, the specification where the expected rate of return to the market is an affine function of the standard deviation corresponds to our assumption in the corollary of a constant ICML, combined with a partial equilibrium argument to ensure that the market portfolio is on the ICML.

An implication of our definition of the investment opportunity set as the ICML is that security returns and their moments may be predictable while the investment opportunity set is constant. Thus, one should be cautious in interpreting empirical evidence of predictability as implying that the investment opportunity set is changing over time.

The corollary potentially provides a theoretical framework for asset pricing tests with time-varying conditional moments which ignore intertemporal hedging premia. For example, Dumas and Solnik (1995) explicitly acknowledge that such tests are inconsistent with the standard theory. The corollary provides a theoretical justification for ignoring the intertemporal hedging premia in conditional asset pricing tests. However, empirical specifications which violate the assumption of a constant (or deterministic) ICML can still not be justified by appeal to our corollary.

7 Conclusions

This study has reexamined the role of intertemporal hedge portfolios in optimal portfolio selection. We showed that only changes in the slope and position of the instantaneous capital market line (ICML) give rise to hedge portfolios. Hedge portfolios that hedge against changes in moments that do not lead to changes in the position and slope of the ICML are superfluous.

This result simplifies the fund separation theorem of Merton (1973).

Because of our finding that investors hedge only against changes in the ICML, we proposed a new definition of the concept of an "investment opportunity set," according to which the investment opportunity set is identical to the ICML. With this definition, predictability in returns does not automatically constitute evidence of a changing investment opportunity set.

It is common in the empirical asset pricing literature to allow for randomly time-varying moments of the returns to securities or to the market portfolio, while ignoring the intertemporal hedging premia that should be present in the specification according to Merton (1973). Our analysis provides a theoretical framework for some of these tests. A constant or deterministic ICML implies that the intertemporal hedging premia disappear. Therefore, they can be ignored in empirical tests whose specification is consistent with a constant or deterministic ICML.

All of our results allow for the type of market incompleteness studied in He and Pearson (1991) and Karatzas et al. (1991). We found that market incompleteness does not matter so long as changes in the position and slope of ICML are driven by a vector of hedgeable state variables, one of which is proportional to the logarithmic portfolio. Market incompleteness does not upset the existence and uniqueness of an optimal portfolio or affect its composition.

Appendix

In this appendix, we prove the proposition and the corollary.

The propsition follows from Lemmas 2 and 3 below. Lemma 1 will be used in the proof of Lemma 2.

Lemma 1 *Under the assumptions of the proposition, $\lambda\sigma^\top b^\top$ is measurable and adapted with respect to F^B, and*

$$\frac{d\Pi}{\Pi} = -r\,dt - \lambda\sigma^\top b^\top\,dB$$

In particular, $\Pi(T)$ is measurable with respect to F^B_T.

Proof: By assumption, there exists a $(1 \times m)$-dimensional process η, measurable and adapted with respect to F^B, such that $\phi^{\ln} = \eta b$. The vector of prices of risk is

$$\lambda = \phi^{\ln}\sigma = \eta b\sigma$$

and so

$$\lambda\sigma^\top b^\top = \eta b\sigma\sigma^\top b^\top = \eta$$

Now,

$$\frac{d\Pi}{\Pi} = -r\,dt - \lambda\,dW = -r\,dt - \eta b\sigma\,dW = -r\,dt - \lambda\sigma^\top b^\top\,dB$$

It follows that Π is an Itô process with respect to B and F^B, and in particular, $\Pi(T)$ is measurable with respect to F^B_T. $\square$

Lemma 2 *Under the assumptions of the proposition, any claim Y^* which is measurable with respect to F^B_T and such that $\Pi(T)Y^*$ is integrable, can be replicated at an initial cost w_0 given by $\Pi(0)w_0 = E(\Pi(T)Y^*)$, by a portfolio strategy $\tilde{\Delta}$ which is a combination of the hedge portfolios.*

Proof: For $k = 1, \dots, m$, let b_{k-} denote the k'th row in b, and let Φ be the m-dimensional column vector valued process whose k'th element Φ_k is the value process of the portfolio strategy b_{k-}, assuming an initial value of $\Phi_k(0) = 1$. Then

$$\begin{aligned} d\Phi &= \mathcal{D}(\Phi)\left[\left(b\sigma\lambda^\top + r\iota\right)\, dt + b\sigma\, dW\right] \\ &= \mathcal{D}(\Phi)\left[(\eta + r, r, \ldots, r)^\top\, dt + dB\right] \end{aligned}$$

The process Φ is adapted and measurable with respect to F^B.

Consider the reduced economy driven by B and F^B, where the risky asset prices are Φ_k, $k = 1, \ldots, m$ and the money market account is the same as in the original economy. In the reduced economy, the vector of prices of risk is $(\eta, 0, \ldots, 0)$, and the state price process is Π, like in the original economy.

Since the reduced economy has dynamically complete markets, there exists a self-financing trading strategy $\bar{\gamma} = (\gamma_0, \gamma)$ in the reduced economy which replicates Y^* and whose value process

$$V = \gamma_0 M + \gamma\Phi$$

has the property that ΠV is a martingale with respect to F^B. Since Y^* and Π are positive, the value process V is positive. Since ΠV is a martingale, in particular,

$$\Pi(0)V(0) = E[\Pi(T)V(T)] = E[\Pi(T)Y^*]$$

and so $V(0) = w_0$.

Now let β be the portfolio strategy in the reduced economy corresponding to the trading strategy $\bar{\gamma}$:

$$\beta = \gamma\mathcal{D}(\Phi)/V$$

and let $\tilde{\Delta}$ be the portfolio strategy in the original economy given by $\tilde{\Delta} = \beta b$. It is then easily seen that if $\tilde{\Delta}$ is started off at the initial wealth level $w_0 = V(0)$, then its value process is V, and, hence, it replicates Y^* in the original economy. $\square$

Lemma 3 *If $Y > 0$ is a claim such that Y is measurable with respect to F_T and $\Pi(T)Y$ is integrable, then there exists a claim $Y^* > 0$, measurable with respect to F_T^B, such that $\Pi(T)Y^*$ is integrable, $E(\Pi(T)Y^*) = E(\Pi(T)Y)$, and $E(Y \mid Y^*) = Y^*$.*

Proof: Simply set $Y^* = E(Y \mid \Pi(T))$. Then Y^* is measurable with respect to $\Pi(T)$, and, hence, $E(Y \mid Y^*) = E[E(Y \mid \Pi(T)) \mid Y^*] = E[Y^* \mid Y^*] = Y^*$. $\square$

The proposition can now be proved from Lemmas 2 and 3, as follows. Let Y be the claim replicated by the portfolio strategy $\tilde{\Theta}$, starting from initial wealth w_0. Let Y^* be a claim as in Lemma 3. By Lemma 2, Y^* can be replicated at initial cost w_0 by a portfolio strategy $\tilde{\Delta}$ which is a combination of the hedge portfolios.

If u is a risk-averse (concave) utility function such that $u(Y^*)$ is integrable above, then it follows from Jensen's inequality that $u(Y)$ is also integrable above, and $Eu(Y) \leq Eu(Y^*)$:

$$Eu(Y) = E[E(u(Y) \mid Y^*)] \leq E[u(E(Y \mid Y^*))] = Eu(Y^*)$$

Therefore, every risk averter with initial wealth w_0 will prefer the portfolio strategy $\tilde{\Delta}$ to the portfolio strategy Θ.

Proof of the corollary:

Set $m = 1$,

$$b = \frac{1}{\sqrt{\lambda\lambda^\top}} \lambda\sigma^\top \left(\sigma\sigma^\top\right)^{-1} = \frac{1}{\sqrt{\lambda\lambda^\top}} \phi^{\ln}$$

Then

$$b\sigma = \frac{1}{\sqrt{\lambda\lambda^\top}} \lambda\sigma^\top \left(\sigma\sigma^\top\right)^{-1} \sigma = \frac{1}{\sqrt{\lambda\lambda^\top}} \lambda$$

and

$$b\sigma\sigma^\top b^\top = \frac{1}{\sqrt{\lambda\lambda^\top}} \lambda\lambda^\top \frac{1}{\sqrt{\lambda\lambda^\top}} \lambda = 1$$

Define B by

$$B(t) = \int_0^t b\sigma \, dW = \int_0^t \frac{1}{\sqrt{\lambda\lambda^\top}} \lambda \, dW$$

Then B is a Wiener process relative to the filtration F, and B is hedgeable with hedge portfolio b. The logarithmic portfolio is proportional to b, since

$$\phi^{\ln} = \sqrt{\lambda\lambda^\top} b$$

Since r and $\sqrt{\lambda\lambda^\top}$ are deterministic, they are adapted and measurable with respect to F^B. The result now follows from the proposition.

$\square$

8 References

1. Bodurtha, J. N. and N. C. Mark: "Testing the CAPM with Time-Varying Risks and Returns," *Journal of Finance* 46 (1991), 1485–1505.

2. Chamberlain, G.: "Asset Pricing in Multiperiod Securities Markets," *Econometrica* 51 (1988), 1305–1323.

3. Constantinides, G. M.: "Admissible Uncertainty in the Intertemporal Asset Pricing Model," *Journal of Financial Economics* 8 (1980), 71–86.

4. Cox, J. C. and C. Huang: "Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process," *Journal of Economic Theory* 49 (1989), 33–83.

5. Dumas, B. and B. Solnik: "The World Price of Foreign Exchange Risk," *Journal of Finance* 50 (1995), 445–479.

6. French, K. R., G. W. Schwert and R. F. Stambaugh: "Expected Stock Returns and Volatility," *Journal of Financial Economics* 19 (1987), 3–29.

7. Harrison, J.M., and D. Kreps: "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory* 20 (1979), 381–408.

8. He, H., and N. D. Pearson: "Consumption and Portfolio Policies with Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case," *Journal of Economic Theory* 54 (1991), 259–304.

9. Karatzas, I., J. Lehoczky, S. E. Shreve and G. Xu: "Martingale and Duality Methods for Utility Maximization in an Incomplete Market," *SIAM Journal of Control and Optimization* 29 (1991), 702–730.

10. Merton, R. C.: "Optimum Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory* 3 (1971), 373–413. Chapter 5 in Merton (1992).

11. Merton, R. C.: "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41 (1973), 867–888. Chapter 15 in Merton (1992).

12. Merton, R.: *Continuous-Time Finance*, Blackwell Publishers, Cambridge, Ma. (1992).

13. Ocone, D. L. and I. Karatzas: "A Generalized Clark Representation Formula, with Application to Optimal Portfolios," *Stochastics and Stochastics Reports* 34 (1991), 187–220.